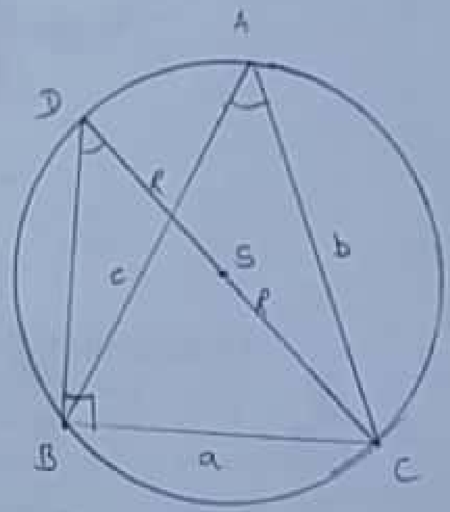
Properties of Triangles

In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$. , R is circumradius. (Sine Rule)

$\angle A$ is acute ($A < 90$)



S → circumcenter, $SD = SC = R$.

$CD = 2R$.

$\angle DBC = 90^\circ$, DBC is Right angle triangle.

Because, semicircle angle is 90°.

→ $\angle BAC = \angle BDC$

Angles is same segment of circle are equal.

$\sin A = \sin \angle BAC = \sin \angle BDC$.

$$\frac{BC}{CD} = \frac{a}{2R}$$

$\boxed{a = 2R\sin A}$ similarly $b = 2R\sin B$, $c = 2R\sin C$.

(ii) $\angle A = 90^\circ$.

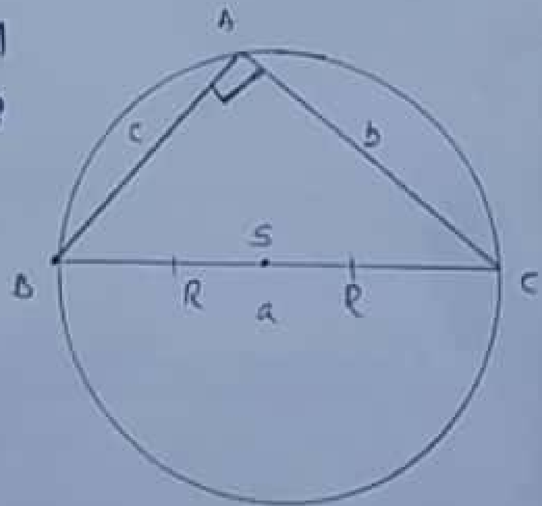
$BC = a = 2R \cdot 1$ $\qquad \sin 90^\circ = 1$

$BC = 2R\sin A$. $\qquad \sin A = 90$

$a = 2R\sin A$

$a = 2R\sin 90^\circ$

$\boxed{a = 2R\sin A}$



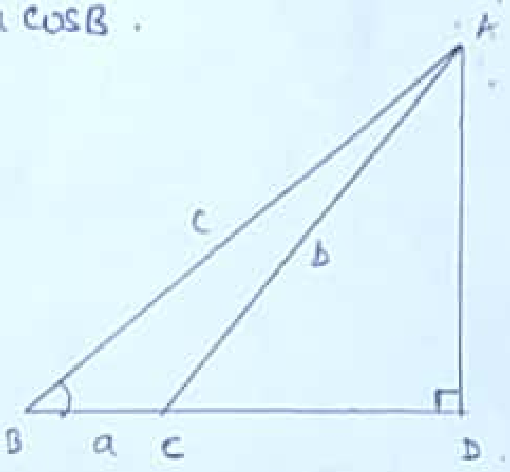
|ly $b = 2R\sin B$

$c = 2R\sin C$

* $a = 2R\sin A$ — a ↓ side 'a'; $\sin A$ ↓ angle opposite to side 'a'.

$b = 2R\sin B$ — b ↓ Side 'b'; $\sin B$ ↓ angle opposite to side 'b'

* Cosine rule! ΔABC, $b^2 = c^2 + a^2 - 2ca\cos B$.



$AD \perp BC$. (BC is extended).

$\cos B = \frac{BD}{AD}$.

Pythagoras theorem, $side_1^2 + side_2^2 = hypotense^2$ in right angle Δle

$AB^2 = BD^2 + AD^2$

$BD = BC + CD$.

$AB^2 = (BC+CD)^2 + AD^2$

$= BC^2 + CD^2 + 2\cdot BC\cdot CD + AD^2$.

$= a^2 + CD^2 + 2BC\cdot CD + AD^2$

Take Δle ADC, $AC^2 = DC^2 + AD^2$

$AC^2 = CD^2 + AD^2 \longrightarrow (1)$

$AB^2 = BC^2 + \underline{CD^2} + 2BC\cdot CD + \underline{AD^2}$

$= BC^2 + AC^2 + 2BC\cdot CD$

$= BC^2 + AC^2 + 2BC(\cos C\cdot AC)$

$CD = \rightarrow$

$\cos C = \frac{CD}{AC}$

$b^2 = a^2 + c^2 + 2\cdot a\cdot \cos C\, b$.

$AB^2 = BD^2 + AD^2$

$= BC^2 + CD^2 + 2\cdot BC\cdot CD + AD^2$

ΔADC, $AC^2 = AD^2 + DC^2$

$= (AB^2 - DB^2) + (BD - BC)^2$

$AD^2 = AB^2 - DB^2$

$\downarrow$

Pythagorean theorem.

$BD = BC + DC$

$DC = BD - BC$

$= AB^2 - \cancel{DB^2} + \cancel{BD^2} + BC^2 - 2BD\cdot BC$

$AC^2 = AB^2 + BC^2 - 2BD\cdot DC$

$$\boxed{b^2 = a^2 + c^2 - 2ac\cos B}$$

Side 'c)

sum of square other side

product of two side

angle opposite to side 'b'.

in ΔABC, $\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$

$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$\therefore$ we have $\cos 2A = ?$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Take $B=A$, $\rightarrow \cos(A+A) = \cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1-\cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1.$

$= 2(1-\sin^2 A) - 1$

$= 2 - 2\sin^2 A - 1$

$= 1 - 2\sin^2 A.$

$\rightarrow \sin^2 A + \cos^2 A = 1$

$\sin^2 A = 1 - \cos^2 A$

$\rightarrow$ $\boxed{\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}}$

$\therefore$ From these, $\cos A = 1 - 2\sin^2 A/2.$

$\therefore 2\sin^2 A/2 = 1 - \cos A.$

$\cos A \rightarrow$ cosine Rule

$b^2 = a^2 + c^2 - 2ac\cos B$

$a^2 = b^2 + c^2 - 2bc\cos A.$

$\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc}}$

$1 - \frac{(b^2+c^2-a^2)}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}.$

$\rightarrow \frac{a^2 - (b^2+c^2-2bc)}{2bc}.$

$(b+c)^2 = b^2+c^2+2bc$

$(b-c)^2 = b^2+c^2-2bc.$

$\rightarrow \frac{a^2 - (b-c)^2}{2bc}$

$(a+b)(a-b) = a^2 - b^2.$

$\rightarrow \frac{(a+b-c)\cdot(a-b+c)}{2bc}$

$a+b+c = 2s, \rightarrow$

$a+b+c-2c = 2s-2c$

$a+b-c = 2(s-c).$

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$a+b+c=2s, \rightarrow a+b-c=2(s-c)$.

$a-b+c=2(s-b)$;

$$1-\cos A = \frac{2(s-c)\cdot(s-b)}{2bc} = \frac{2(s-b)(s-c)}{bc} = 2\sin^2 A$$

$$\boxed{\sin A/_2 = \sqrt{\frac{(s-b)(s-c)}{bc}}}$$

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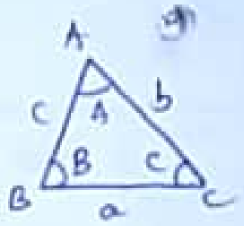
→ $\cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$

$\cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$

$\cos A = 2\cos^2 A/2 - 1$



$1+\cos A = 2\cos^2 A/2$

$$1+\cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{(\cancel{2}s)\cdot(2(s-a))}{\cancel{2}bc}$$

→ $b^2 = c^2+a^2-2ac\cos B$

$a^2 = b^2+c^2-2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

→ $(b+c)^2 = b^2+c^2+2bc$

→ $(b+c)^2-a^2 = (b+c+a)(b+c-a)$

→ $a+b+c = 2s$

(s → semi perimeter)

$a+b+c-2a = 2s-2a$

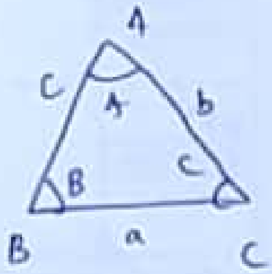
$b+c-a = 2(s-a)$.

$$\cancel{2}\cos^2 A/2 = \frac{\cancel{2}s(s-a)}{bc}$$

$$\boxed{\cos A/2 = \sqrt{\frac{s(s-a)}{bc}}}$$

$$\rightarrow \tan A/2 = \frac{\sin A/2}{\cos A/2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \Big/ \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$



→ $\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\sin B/2 = \sqrt{\frac{(s-a)(s-c)}{ac}}$

$\sin C/2 = \sqrt{\frac{(s-a)(s-b)}{ab}}$

sin(half of angle)

$$= \sqrt{\frac{(\text{semiperimeter} - \text{adjacent side to angle})(\text{semiperimeter} - \text{adjacent side(2) to angle})}{\text{adjacent } s_1 \text{ side} \times \text{adjacent side(2)}}}$$

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$$\cos A/2 = \sqrt{\frac{s(s-a)}{bc}} \qquad \tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\cos B/2 = \sqrt{\frac{s(s-b)}{ac}} \qquad \tan B/2 = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\cos C/2 = \sqrt{\frac{s(s-c)}{ab}} \qquad \tan C/2 = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Also. $\sin A = 2\sin A/2 \cos A/2$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$

[$\sin(A+A)$
$= \sin A\cos A + \cos A\sin A$
$\sin 2A = 2\sin A\cos A$
$\sin A = 2\sin A/2 \cos A/2$.

$$\sin A = \frac{2}{(\sqrt{bc})^2}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

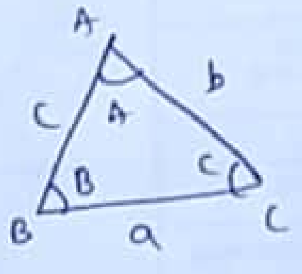
$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$\sqrt{s(s-a)(s-b)(s-c)}$ = Area of ΔABC.

$$\Delta = \frac{1}{2}\sin C\, ab$$

$$\text{are}(\Delta) = \frac{1}{2}ab\sin C.$$

[product of two sides, and sine angle b/w them)



$$\rightarrow \boxed{\begin{aligned} \text{ar}(\Delta^{le}) &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}ac\sin B \\ &= \frac{1}{2}.bc\sin A. \end{aligned}}$$

Also $\rightarrow$ $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}\times(2R\sin A)(2R\sin B)c\sin C)$

since $\sin$ Rule $= a/\sin A = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\boxed{\Delta^{k} = 2R^2\sin A\sin B\sin C}$$

$\Delta = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\Delta = \frac{1}{2}ab\left(\frac{c}{2R}\right)$

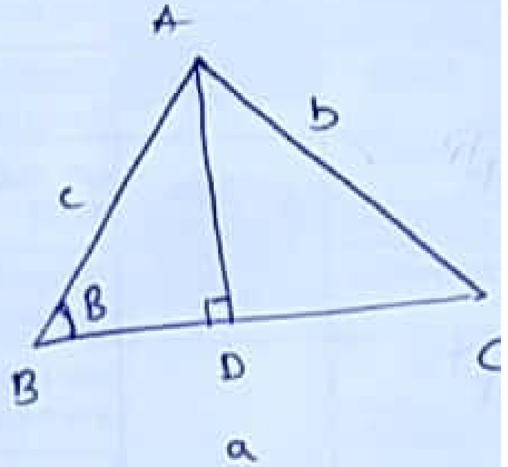
$\frac{c}{2R} = \sin C$

$\Delta = \frac{1}{4}\frac{abc}{R}$

$$\boxed{\Delta = \frac{abc}{4R}}$$

$\Delta = ar(\triangle ABC) = \frac{1}{2}bc\sin A$

Draw $AD \perp BC$.



$\Delta = \frac{1}{2}b \times h = \frac{1}{2} \times BC \times AD$

$= \frac{1}{2} \times a \times AD$

$\Delta = \frac{1}{2} \times a \times c\sin B$

$\sin B = \frac{AD}{AC} = \frac{AD}{c}.$

$$\boxed{\Delta = \frac{1}{2}ac\sin B}$$

Tip: $\frac{1}{2}$ (side$_1$) $\times$ (side2) $\times$ Sin(angle b/w there side1 & side2)

$\Delta = \frac{1}{2}ab\sin C$, (a, b) ← two sides, C° → b/w a and b.

$= \frac{1}{2}bc\sin A$ b, c are sides, A° b/w side b and c

$= \frac{1}{2}ac\sin B$ a, c two sides, B° b/w side a and c

→ $\Delta = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin B$

Tan A/2

$$\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos A/2 = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan A/2 = \frac{\sin A/2}{\cos A/2} = \left[\sqrt{\frac{(s-b)(s-c)}{bc}}\right] \Big/ \left[\sqrt{\frac{s(s-a)}{bc}}\right]$$

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{(s-b)(s-c)}{(s-b)(s-c)}}$$

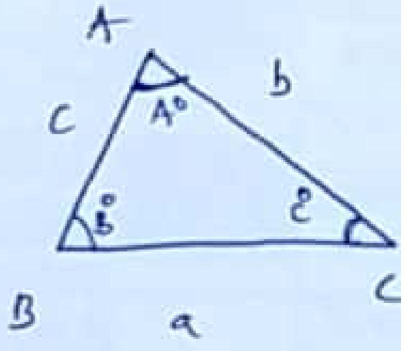
$$= \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta}$$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)} \rightarrow$ Heron's formulae.

$$\tan A/2 = \frac{(s-b)(s-c)}{\Delta}$$

A B C $\rightarrow$ angles
a, b, c $\rightarrow$ sides.

$$\tan B/2 = \frac{(s-c)(s-a)}{\Delta}$$



$$\tan C/2 = \frac{(s-a)(s-b)}{\Delta}$$

Tip: Tan of half of an angle

$$= \frac{(\text{Semiperimeter} - \text{side1})(\text{Semiperimeter} - \text{side2})}{\text{area of } \Delta^{le}}.$$

angle b/w side1 and side2 is half Angle.

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$\rightarrow \tan A/_2 = \sqrt{\dfrac{s(s-b)(s-c)}{s(s-a)}} \times$

$= \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\dfrac{s(s-a)}{s(s-a)}} = \sqrt{\dfrac{s(s-a)(s-b)(s-c)}{[s(s-a)]^2}}$

$= \dfrac{\Delta}{s(s-a)}$

$\tan A/_2 = \dfrac{\Delta}{s(s-a)} = \dfrac{(s-b)(s-c)}{\Delta}$

Tip $\rightarrow$ $\Delta^2 = s(s-a)(s-b)(s-c)$.

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

By remembering this simple formulae of Heron's, and just rearranging term, we can find $\tan A/_2$.

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $\Delta^2 = s(s-a)(s-b)(s-c)$

$$\boxed{\dfrac{\Delta}{s(s-a)} = \dfrac{(s-b)(s-c)}{\Delta} = \tan A/_2}$$

$\tan B/_2 = \dfrac{(s-c)(s-a)}{\Delta} = \dfrac{\Delta}{s(s-b)}$

$\tan C/_2 = \dfrac{(s-a)(s-b)}{\Delta} = \dfrac{\Delta}{s(s-c)}$

Tip.

$\left[\tan A/_2 = \dfrac{1}{\cot A/_2} \right]$

$\ast \rightarrow$ 'Mollweide' rules.

In ΔABC, $\dfrac{a+b}{c} = \dfrac{\cos[(A-B)/2]}{\sin C/2}$.

$\because a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$.

$$\frac{a+b}{c} = \frac{2R\sin A + 2R\sin B}{2R\sin C} = \frac{\sin A + \sin B}{\sin C}$$

$$= \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin C/2 \cos C/2} = \frac{\sin(90 - C/2)\cos\left(\frac{A-B}{2}\right)}{2\sin C/2 \cos C/2}$$

$A+B+C = 180$

$\frac{A+B}{2} + C/2 = 90$

$C/2 = 90 - \frac{(A+B)}{2}$

$$\therefore \frac{\cos\left(\frac{A-B}{2}\right)}{\sin C/2}$$

$$\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin C/2}.$$

$$\frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin A/2}.$$

$$\frac{a+c}{b} = \frac{\cos\left(\frac{C-A}{2}\right)}{\sin B/2}$$

Imp.

Incircles, Excircles of Δ^{le}.

→ Circle touches three side ΔABC → incircle.

incenter → I, inradius → r.

→ point of concurrence of internal bisector of angles of Δ^{le}, is incenter (I).

In ΔABC, $\Delta = rS$.

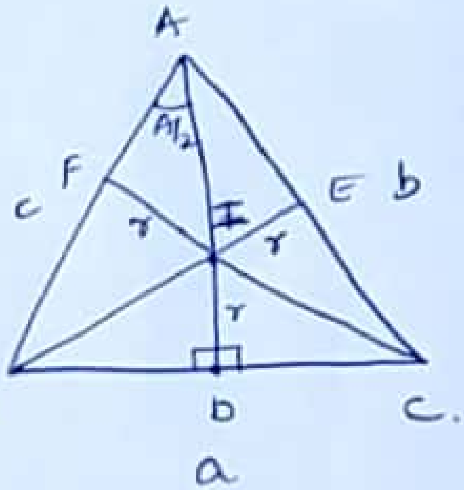
(1) $\Delta = rS$

I incenter of angular bisector meet.

I → concurrence point.

Draw $ID \perp BC$, $IE \perp CA$, $IF \perp AB$.

$ID = IE = IF = r$.



$$ar(\Delta^{le} ABC) = ar(\Delta BIC) + ar(\Delta CIA) + ar(\Delta AIB)$$

$ar(\Delta BIC) = \frac{1}{2} \times b \times h = \frac{1}{2} \times (BC) \times (ID) = \frac{1}{2} \times a \times r$

$ar(\Delta CIA) = \frac{1}{2} \times b \times h = \frac{1}{2} \times (AC) \times IE = \frac{1}{2} \times b \times r$

$ar(\Delta AIB) = \frac{1}{2} \times b \times h = \frac{1}{2} \times (AB) \times IF = \frac{1}{2} \times c \times r$

$\Delta = ar(\Delta BIC) + ar(\Delta AIC) + ar(\Delta AIB)$

$\Delta = \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr$

$\Delta = \frac{1}{2} r(a+b+c)$. $\qquad [a+b+c = 2S]$

$\Delta = \frac{1}{\cancel{2}} \times (r) \times (\cancel{2}S)$

$\boxed{\Delta = rS}$

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